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Relaxation in interacting arrays of oscillators

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We analyze a system of interacting arrays of globally coupled nonlinear oscillators. The relaxation in the interacting arrays with different interaction strengths is compared to that in an array not subject to interaction with others. The relaxation of the latter is found to be an exponential function of time. On the other hand the relaxation of the interacting arrays is slowed down and departs from an exponential of time. There exists a crossover time, t_c, before which relaxation of the interacting arrays is still an exponential function. However, beyond t_c relaxation is no longer exponential but well approximated by a stretched exponential $\exp[-(t/\tau)^{\beta}]$. The fractional exponent β decreases further from unity with increasing interaction strength. The result bears strong similarity to the basic features suggested by the coupling model and seen experimentally by neutron scattering for relaxation in densely packed interacting molecules in glass-forming liquids. [S1063-651X(96)51109-8]

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I. INTRODUCTION

Recent results [1] have shown that studies of nonlinear dynamical systems can enhance the understanding of some fundamental problems in physics, such as stability of the solar system, phase transitions, turbulence, and the ergodic problems in statistical mechanics. One common characteristic of such systems is the irreversibility of the dynamics due to sensitive dependence on initial conditions. Lately in the community of research in chaos, there is considerable interest in nonlinearity coupled oscillators. Various models of a globally coupled oscillator array [2–6] have been studied. It was conjectured and later proved (though with conditions) that for coupling strength below a certain threshold, such systems would relax to an incoherent state [7,8]. Recently, the relaxation process in such an array is found to be exponentially fast [9].

It would be of interest to study the relaxation of a more complex system, consisting of a number of such arrays coupled by nonlinear interactions. Such a study may be beneficial to the understanding of relaxation processes in glass-forming viscous liquids, polymers, and ionic conductors, to name a few. These problems in condensed matter physics, physical chemistry, and materials science involve irreversible processes in densely packed interacting systems [10]. The interactions in these systems come from nonlinear potentials such as that of Lennard-Jones in polymers and of Coulomb in vitreous fast ionic conductors. What we learn from the solution of this problem should be promising to shed some light on the nonlinearity coupled many-body problem.

II. GLOBALLY COUPLED OSCILLATORS

We consider a simplified version of an array of coupled oscillators studied by Strogatz, Mirollo, and Matthews [9]:

$$\frac{d}{dt}\varphi_i=\frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_j-\varphi_i),$$

for $i,j=1,\ldots,N\ll 1$, which is closely related to the equations of motion studied by Strogatz, Mirollo, and Matthews [9]. The problem was originally motivated by the study of the biological phenomenon of mutual synchronization [7]. We further simplify the problem to a map by picking appropriate time steps and rescaling the time. The system becomes a map,

$$\varphi_i'=\varphi_i+\frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_j-\varphi_i)\,. \tag{1}$$

Like Strogatz, Mirollo, and Matthews we are interested in the decay of the phase coherence, r, which is the absolute value of the order parameter, $r\exp(i\psi)$, defined by

$$r=|re^{i\psi}|=\left|\frac{1}{N}\sum_{j} e^{i\varphi_j}\right|. \tag{2}$$

We find that our array of coupled oscillators decays to an incoherence state, i.e., $r=0$ for $K<0$. Following the argument of Strogatz, Mirollo, and Matthews [9], the relaxation is exponential [11]. Numerical calculation shows that the decay of r is almost exactly an exponential function of time at least up to the longest time we have carried out so far and 99.9% of the initial value of r has decayed (see Fig. 1). This resembles the (Debye) relaxation of an isolated molecule in dilute solution which is also exponential.

III. SYSTEMS OF INTERACTING ARRAYS

Now if we allow a number of these arrays to interact to obtain an even more complex system, the interesting problem is to find out what modification these interactions will have on the relationship of r for each array towards incoherence. The interactions between the arrays $\alpha=1,\ldots,M$ can be chosen in several ways. One choice we have made is indicated by the new maps,

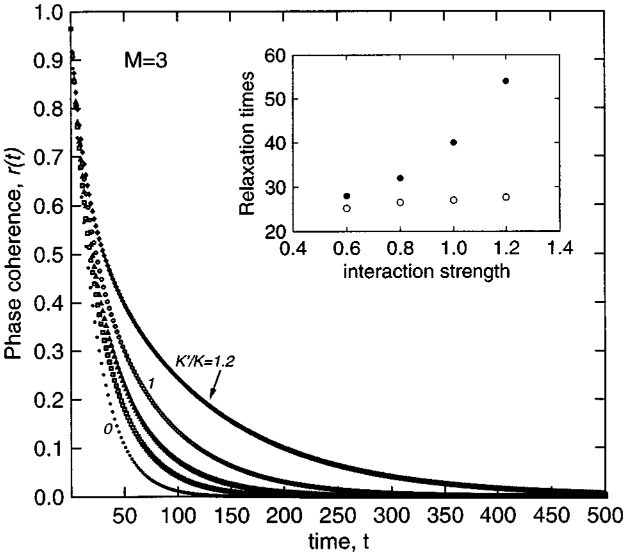

FIG. 1. Decay of phase coherence, $r(t)$, calculated numerically $M=3$, $K=-0.03$ and interaction strength $K'/K=0$ (closed circle), 0.6 (square), 0.8 (triangle), 1.0 (open circle), and 1.2 (diamond). The calculated $r(t)$ for $K'/K=0$, corresponding to a single array with dynamics described by Eq. (1), conforms well to the exponential decay. The inset shows the relaxation time τ_0 (open circles) and τ^* (closed circles), as functions of interaction strength.

$$\varphi'_{i\alpha}=\varphi_{i\alpha}+\frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_{j\alpha}-\varphi_{i\alpha})$$
$$-\frac{K'}{MN}\sum_{\beta=1}^{M}\sum_{j=1}^{N}\sin(\varphi_{j\beta}-\varphi_{i\alpha}). \tag{3}$$

The strength of the interactions between the arrays is measured by K'/K. The interacting arrays mimic an assembly of molecules densely packed together.

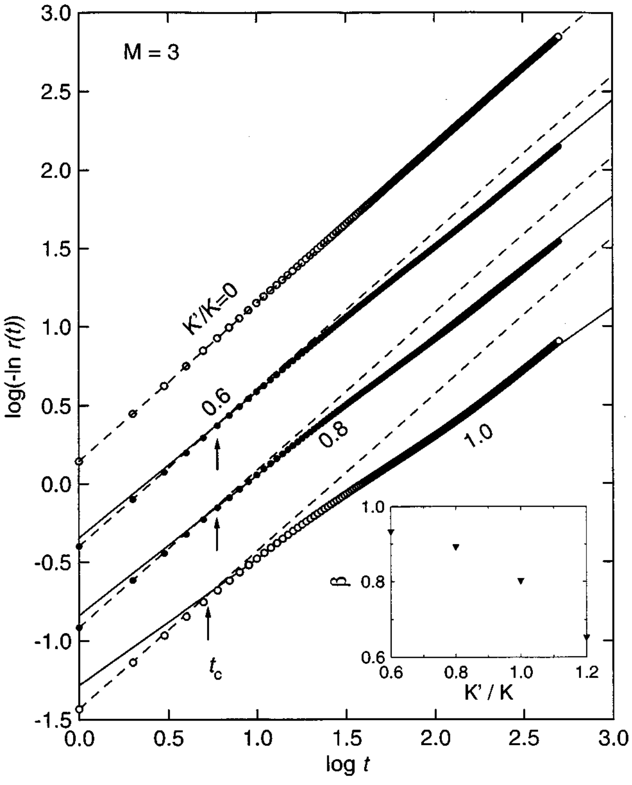

FIG. 2. Log$[-\ln r(t)]$ vs log$t$, calculated numerically for $K=-0.03$, $M=3$, and interaction strength $K'/K=0$, 0.6, 0.8, and 1.0. The dashed lines are the exponential fits and the solid are the stretched exponential fits. Curves for $K'/K=0$, 0.6, and 0.8 are shifted vertically up in multiples of 0.5 to avoid overlapping on the graph. The crossover time t_c is indicated by a vertical arrow. The calculated $r(t)$ for other interaction strengths conforms well to the $\exp(-t/\tau_0)$ for $t<t_c$ and the $\exp[-(t/\tau^*)^\beta]$ for $t>t_c$. The inset shows the Kohlrausch exponent β as a function of interaction strength. The exponent β for $K'/K=1.2$ case is also included in the inset.

IV. NUMERICAL RESULTS: RELAXATION OF PHASE COHERENCE

We have iterated the maps to obtain the evolutions of the coupled array system numerically. From the result we calculate the decay of the phase coherence r for each array. Figure 1 shows that the decay of r is slowed down by the interactions between arrays. By inspection (and as shown in the inset) it is clear that the degree of slowing down varies directly as the interaction strength K'/K, which has been chosen to have the values of 0.6, 0.8, 1.0, and 1.2. In all these calculations, $M=3$ and N is fixed at the value of 32. The results shown in Fig. 1 are calculated for $M=3$. The results are consistent with an initial exponential decay $\exp(-t/\tau_0)$, and crossover at some t_c to a stretched exponential decay, $\exp[-(t/\tau^*)^\beta]$, as indicated by the curves drawn in Fig. 2. The Kohlrausch exponent [12] β is actually the slope of the solid line fits in Fig. 2. Fitting the calculated results this way we find that the relaxation time τ^* increases and the exponent β decreases with the interaction strength, K'/K, between the arrays (see insets of Fig. 1 and 2). These behaviors are in accord with the results of the coupling model [13] for relaxation of densely packed interacting molecular systems. Continuity of the two pieces, $\exp(-t/\tau_0)$ and $\exp[-(t/\tau^*)^\beta]$ at t_c guarantees the validity of the relation between τ^* and τ_0 given by

$$\tau^*=[t_c^{-n}\tau_0]^\beta \tag{4}$$

which is the same as

$$\tau^*[(1-n)\omega_c^n\tau_0]^{1/(1-n)} \tag{5}$$

written down in the original version of the coupling model, provided the identifications

$$t_c=(1-n)^{1/n}\omega_c^{-1} \tag{6}$$

and $\beta=1-n$ are made. The reciprocal of the ω_c in Eq. (5) is the time at which the relaxation rate defined by $-(1/C(t))dC(t)/dt$ are equal for the two pieces of correlation functions, $C(t)=\exp(-t/\tau_0)$ for $t<t_c$ and $\exp[-(t/\tau^*)^{1-n}]$ for $t<t_c$; t_c is the time at which they are continuous. Figure 2 shows that t_c decreases with increasing interarray interaction strength. Figure 2 also shows that t_c (indicated by arrows) decreases with increasing interarray

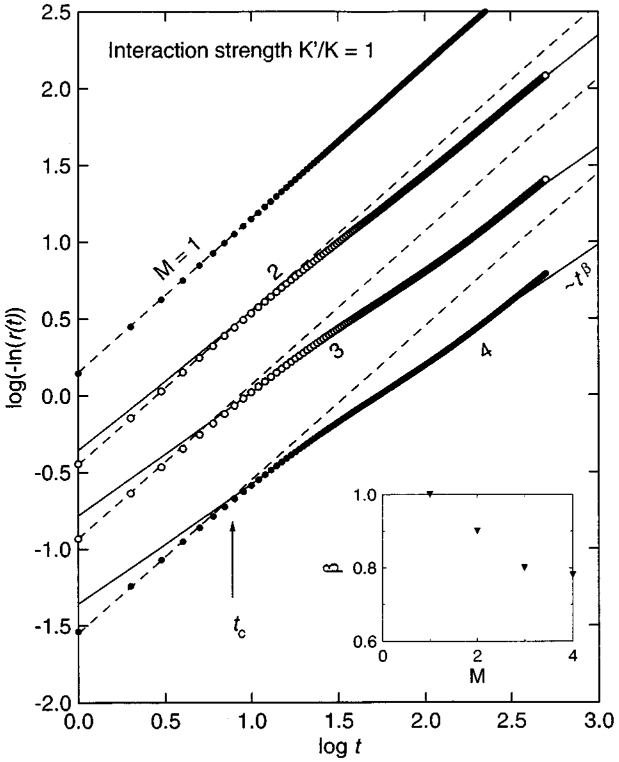

FIG. 3. Log$[-\ln r(t)]$ vs log$t$, calculated numerically for $K=-0.03$, interaction strength $K'/K=1$, and different M's. Curves for $M=1$, 2, and 3 are shifted similarly to curves in Fig. 2. The calculated $r(t)$ for $M=1$, actually the same as for $K'/K=0$ in Fig. 2, corresponding to a single array with dynamics described by Eq. (1), conforms well to the exponential decay. The calculated $r(t)$ for other M's conforms well to the $\exp(-t/\tau^*)$ for $t<t_c$ and the $\exp[-(t/\tau^*)^{\beta}]$ for $t>t_c$. The inset shows β as a function of M.

interaction strength. Thus the relaxation of the nonlinearity coupled arrays of coupled nonlinear oscillators has verified the basic features of the coupling model. The degree of slowing down in relaxation due to the interarray interaction depends also on M, increasing with M but leveling off for $M>4$. Figure 3 shows the results for different M's. Note that $M=1$ is the case without any interaction.

The good correspondence between relaxation of simpler prototype chaotic systems and the key features of the coupling model shows that a theory of the coupling model is in the making. The encouraging results obtained will provide impetus for future study of more complex chaotic Hamiltonian systems that bear closer resemblance to polymers, small molecule glass-forming liquids, and vitreous ionic conductors.

V. CONCLUSION AND DISCUSSION

The effect of interactions between M arrays of coupled nonlinear oscillators has been studied. It is found that for sufficiently large t, $r\sim\exp(-t/\tau)$ for $M=1$, indicating that the coupled array has exponential relaxation. For other $M>1$ with nonzero interaction strength, it is found that there exists a time t_c such that r is still an exponential function of time, $\exp(-t/\tau_0)$, for $t<t_c$, but $r\sim\exp[-(t/\tau^*)^{\beta}]$ for $t>t_c$. Thus the relaxation of the interacting arrays proceeds with a stretched exponential time dependence for $t>t_c$ and $t^*\gg\tau$. Thus the interaction between the arrays is observed to slow down the relaxation of single arrays. As expected, the fractional exponent β decreases further from unity with increasing nonlinearity or larger interaction strengths.

In the study of relaxation in real systems in physics, chemistry, and materials science, it is found that a system without interactions (e.g., an isolated molecule in dilute solution) usually relaxes exponentially. However, when such systems are densely packed and interacting with each other, the relaxation proceeds differently and exhibits many fascinating properties. A coupling model has been very successful in explaining these properties [13]. This model is based on the hypothesis that an interacting system relaxes initially exponentially until a temperature-independent microscopic time, t_c, but relaxes stretched-exponentially afterwards with continuity of the correlation function at the time of crossover [14,15]. A recent neutron scattering measurement on a polymer has shown direct experimental evidence for this hypothesis [16]. In our present work, the addition of interarray interactions introduces additional nonlinearity which has similar effects on the relaxation towards equilibrium as many-body interactions have on relaxation in densely packed molecular systems. Since the numerical result obtained bears strong similarity to the basic features suggested by the coupling model [14] and seen in neutron scattering experiment [16], the interacting array model provides a useful first step in applying nonlinear dynamical models to the study of irreversible processes of real physical systems in physics, chemistry, and materials science.

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